Stochastic resonance of a subdiffusive bistable system driven by Lévy noise based on the subordination process

2009 J. Phys. A: Math. Theor. 42 475003

(http://iopscience.iop.org/1751-8121/42/47/475003)

J. Phys. A: Math. Theor. **42** (2009) 475003 (12pp) doi:10.1088/1751-8113/42/47/475003

Stochastic resonance of a subdiffusive bistable system driven by Lévy noise based on the subordination process

Lei Zhang, Aiguo Song and Jun He

School of Instrument Science and Engineering, Southeast University, Sipailou 2, Nanjing 210096, People's Republic of China

E-mail: topo.zhang@gmail.com

Received 1 July 2009, in final form 5 October 2009
Published 6 November 2009
Online at stacks.iop.org/JPhysA/42/475003

Abstract

The stochastic resonance of a subdiffusive bistable system driven by Lévy noise with an input of sinusoidal signal is studied. We employ the subordination technique to model the subdiffusive system with a time-space-dependent external driving force. It is shown that stochastic resonance is robustly present in the competitive case, but is reduced by the lower subdiffusive index α or the lower superdiffusive index μ. However, when the system parameters are optimally configured, long jumps induced by Lévy flights may enhance the stochastic resonance phenomenon, and in a certain range, the subdiffusive effect can be cancelled out by superdiffusive dynamics. The subordination technique provides a possible physical insight into the stochastic resonance phenomenon.

PACS numbers: 05.04.−a, 02.50.−r

1. Introduction

Addition of noise can play an active role in enhancing the nonlinear system output, and the counterintuitive phenomenon is termed stochastic resonance (SR). In the past two decades, SR has been widely studied in a variety of fields, both in theory and in application [1, 2]. However, almost all research studies treated input noise as solely Gaussian and hence, had finite variance. Since non-Markovian SR theory emerged [3, 4], SR of anomalous diffusion has attracted much attention, and particularly the pioneering works by Hänggi *et al* [4] showed that long waiting time induced by heavy-tailed distribution or long-length jump of Lévy noise should be considered more carefully, based on the SR viewpoint. Recently, some works have been carried out to investigate the quantities which are related to SR in anomalous diffusion. For example, linear response theory of the subdiffusive bistable system was studied by Liu *et al* [5]. It is shown that strong subdiffusive dynamics can weaken input–output synchronization and make the SR effect become less pronounced. Parameter-induced aperiodic stochastic resonance (ASR) of the subdiffusive bistable system was also discussed by Xu *et al* [6]. It is shown that with optimized system parameters, the SR effect decreases with the increase of subdiffusive character, while for the superdiffusive case, Dybiec *et al* [7] proved that periodic SR still might occur in the presence of symmetric or skewed Lévy noise. Compared with Gaussian noise at the same intensity, the SR effect is weakened with the increase of superdiffusive index. Parameter-induced ASR of a bistable system driven by Lévy noise was also studied by Zeng *et al* [8]. It is shown that under optimal system parameters the parameter-induced ASR effect was enhanced as spatial non-locality is increased. Due to universality of subdiffusion or superdiffusion, the anomalous diffusion effect of SR has become an interesting topic [9, 10]. However, the study of SR in the coexisting phenomenon between subdiffusion and superdiffusion has received limited attention. To our knowledge, the SR effects of the subdiffusive bistable dynamics system driven by Lévy noise have not been investigated. It is worth studying the cooperative effects between subdiffusive dynamics and Lévy noise from the SR viewpoint.

In contrast with recent works, our current attention is on analysing the SR effect of the subdiffusive bistable system driven by Lévy noise with an input of sinusoidal signal. The paper is organized as follows. In section 2, some recent related fractional Fokker–Planck equation (FFPE) technologies are reviewed briefly. It is shown that a valid FFPE form with time-space-dependent force field is not easily available. The subdiffusive bistable dynamics driven by Lévy noise is modelled by the subordination technique based on the generalized Langevin equation. The numerical method is presented. In section 3, the anomalous properties of the subdiffusive bistable system are studied by tuning noise intensity and changing system parameters. System output signal–noise-ratios (SNRs) are computed, and the results are discussed. We show that the SR is robust in the competitive case. Finally, the conclusions are made in section 4.

2. Subdiffusive bistable model driven by Lévy noise

The FFPE has played an important role in studying the dynamic properties of anomalous diffusion [11]. Some recent related technologies are reviewed based on the SR viewpoint. In the pure subdiffusive case, the subdiffusive FFPE with double-well potential is solved numerically by the eigenfunction expansion method [5]. The solution of the FFPE with a harmonic potential is presented in Hermite polynomial form [11]. The FFPE with arbitrary external potential is solved by the state-dependent diagonalization method [12]. It is also shown that the eigenfunction expansion problem of the FFPE can be transformed to the variational problem and then it can be solved by the finite element method, and an efficient numerical method is also given, which is applicable for an FFPE with any potential [6].

For the pure superdiffusive case, the FFPE in the cases of constant force and linear force has been discussed [11]. However, to our knowledge, analytical solutions of spatial the FFPE with any nonlinear external force field remain a challenging problem. Recently, the FFPE with a bistable potential was solved by employing the Grünwald–Letnikov scheme [8]. The numerical method has become an important tool in studying the SR phenomenon driven by Lévy noise.

The competition model between subdiffusion and superdiffusion can be described by an FFPE with temporary and spatial derivatives

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{V'(x)}{\eta} + D\nabla^{\mu}\right]p(x,t), \qquad (1)$$

which is derived from the continuous time random walk (CTRW) model [11]. Here the parameter D means the diffusion coefficient. The random walker moves with step lengths chosen from a given probability density function (PDF). In the walking process, the walker visits the sites at times chosen from the infinite mean PDF. The long-tailed waiting times (temporal $-t^{-(1+\alpha)}$, $0 < \alpha < 2$) cause slowly decaying memory effects which give rise to temporal derivative or subdiffusion. When step lengths obey a long-tailed distribution ($\sim|x|^{-(1+\mu)}$, $0 < \mu < 2$), corresponding to Lévy flights, this can lead to spatial derivatives or superdiffusion. The operator

$$D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s \tag{2}$$

is the fractional derivative of the Riemann–Liouville type and the operator ∇^{μ} denotes the Riesz fractional derivative. However, Heinsalu and Hänggi *et al* [13] argued that equation (1) is invalid for pure time-dependent force fields, which fails to correspond to the underlying CTRW. A subdiffusive fractional Fokker–Planck dynamics in the class of dichotomously alternating forces is derived,

$$\frac{\partial p(x,t)}{\partial t} = \left[\frac{\partial}{\partial x} \frac{V'(x,t)}{\eta} + D\nabla^{\mu}\right]_0 D_t^{1-\alpha} p(x,t). \tag{3}$$

At present, the correctness of the FFPE (3) is still difficult to validate when extended *ad hoc* to an arbitrary time-space-dependent force field different from the dichotomous case. For related physical arguments, refer to [14]. The existence of any physically valid FFPE with arbitrary potential fields still remains intrigued.

In recent paper, a model based on the Langevin equation and subordination technique has been proposed, and Magdziarz *et al* [15] proved that the PDF of the subordination Langevin process was equivalent to the solution of the FFPE. The authors also tackled the problem of modeling subdiffusion with an arbitrary time-space-dependent driving force [16]. The subordination technique allows us to study the subdiffusive bistable system without referring to the FFPE.

2.1. Model

Since an FFPE describing subdiffusive dynamics with arbitrary time-space-dependent force is still lacking, we overcome the gap by using the subordination process and Langevin equation to model the SR of the subdiffusive bistable system driven by Lévy noise, with an input of sinusoidal signal. This is a main technical difference from recent papers, which are based on the FFPE. The subordinated Langevin equation can be expressed as

$$Y(t) = X(S_t). \tag{4}$$

The subordinator S_t, called the inverse time α-stable process subordinator, $0 < \alpha < 1$, is defined as

$$S_t = \inf\{\tau, U(\tau) > t\} \tag{5}$$

where $U(\tau)$ denotes a strictly increasing α-stable Lévy motion. The parent process $X(\tau)$ is the solution of the Itô stochastic differential equation (SDE)

$$\mathrm{d}X(\tau)/\mathrm{d}\tau = -V'(X(\tau), U(\tau))\eta^{-1}\,\mathrm{d}\tau + D^{1/\mu}\,\mathrm{d}L_{\mu,\beta}(\tau) \tag{6}$$

which is driven by symmetric Lévy noise $L_{\mu,\beta}(\tau)$. We consider the subdiffusive SR system with the time-space-dependent bistable force field

$$F(x,t) = -V'(x,t) = ax - bx^3 + A_0\cos(Qt + \varphi) \tag{7}$$

Assuming that $\varphi = 0$, for the time-space-dependent force $F(x, t)$, it should be noted that after subordination $X(S_t)$, the actual force is given as $F(x, U(S_t))$. By employing the relation

$$U(S_t) = \begin{cases} t & t = t_j \\ t + \Delta L_t & t \neq t_j, \end{cases} \tag{8}$$

where t_j, $j \in N$, is the instant time when a test particle is released from a jth trap, and ΔL_t is random leapover time, it is shown that the particle may be biased by the force $F(x, t)$ [16]. That is to say, we only subordinate the process $X(\tau)$ without subordinating the time-dependent force. The relation plays an important role in studying the subdiffusive bistable dynamics.

The process $X(\tau)$ is indexed by the internal time τ which is not physical time. The subordinator S_t plays a role in helping the subordination parent process $X(\tau)$ change time scale from random internal time τ to physical time t. The process S_t and $X(\tau)$ are assumed to be statistically independent. The role of the subordinator S_t is analogous to the role of the fractional Riemann–Liouville derivative in the FFPE (3) and may be used to describe the heavy-tailed waiting time between successive jumps of a particle. The process S_t is responsible for the subdiffusive behaviour of the system, whereas the parent process $X(\tau)$ introduces Lévy flights behaviour. Therefore, the long rest of the particle induced by subdiffusive dynamics can be characterized by the subordinator S_t, and the long jumps of superdiffusion can be characterized by the Lévy noise $L_{\mu,\beta}(\tau)$. The subordinated process combines both these characteristics, which results in the competition between subdiffusion and Lévy noise [16]. Hence, we may use the subordination technique to explore the competitive model without referring to the FFPE.

Along the paper, the PDFs of Lévy stable noise $L_{\mu,\beta}$ are expressed in terms of characteristic function $\lambda_{\mu,\beta}\,(\kappa;\sigma,\tau)$ [7]

$$\lambda_{\mu,\beta}(x;\sigma,\tau) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{-\mathrm{i}kx}\lambda_{\mu,\beta}(\kappa;\sigma,\theta), \tag{9}$$

where

$$\lambda_{\mu,\beta}(\kappa;\sigma,\theta) = \exp\left[-\theta|k|^{\mu}\left(1 - \mathrm{i}\beta\frac{\kappa}{|\kappa|}\omega(k,\mu)\right) + \mathrm{i}\sigma\kappa\right] \tag{10}$$

and

$$\omega(k,\mu) = \begin{cases} \tan\frac{\pi\mu}{2} & \text{if} \quad \mu \neq 1, \\ -\frac{2}{\pi}\ln|\kappa| & \text{if} \quad \mu = 1. \end{cases} \tag{11}$$

The exponent μ is the Lévy stable index, β is the skewness parameter, θ is the scale parameter and σ is the location parameter.

2.2. *Response measures*

Since the discovery of the SR phenomenon, several different measures which characterize it have been introduced in the literature [17, 18]. Some examples are as follows. In the periodic SR case: (i) output SNR, (ii) the spectral factor, (iii) residence time distribution. In the ASR case: (i) signal detection probability, (ii) information theory-based tools, etc. Throughout this paper, we use the output SNR at the driving frequency:

$$\mathrm{SNR} = 2\left[\lim_{\Delta\omega\to 0}\int_{Q+\Delta\omega}^{Q+\Delta\omega} S(\omega)\,\mathrm{d}\omega\right]\bigg/ S_N(Q). \tag{12}$$

Here

$$\int_{Q-\Delta\omega}^{Q+\Delta\omega} S(\omega)\,\mathrm{d}\omega \tag{13}$$

represents the power carried by the signal and $S_N(Q)$ represents the power of the background noise.

2.3. Numerical method

Next we start to approximate the process S_t and $X(\tau)$ on the lattice $\{t_i = \mathrm{i}\Delta t\colon \mathrm{i} = 0, 1, 2, \ldots, N\}$, where $\Delta t = T/N$ and T is the time horizon. Recall that $X(\tau)$ is given by SDE (6) with Lévy noise, and S_t is the inverse time α-stable process subordinator. In step 1, we simulate the trajectory of the process $U(\tau)$ and its inverse S_t. In step 2, we simulate the trajectory of the process $X(\tau)$ using the standard Euler scheme. Note that the force in equation (7) is equal to $F(X(\tau), U(\tau))$ not $F(X(\tau), \tau)$. Therefore, in step 1, one had to simulate $U(\tau)$ first. Finally, we evaluate the final process $Y(t) = X(S_t)$ by subordinating $X(\tau)$ from step 2 by S_t from step 1. The case is delicate, and one should be careful of choosing an appropriate step length for the lattice. The algorithm of simulating trajectories for the case of time-space-dependent force is as follows.

(I) Our first objective is to approximate the value $S_{t0}, S_{t1}, S_{t2}, \ldots, S_{t\mathrm{N}}$ of the subordinator. To begin with approximation of a realization of the strictly increasing α-stable Lévy motion $U(\tau)$ on the mesh $\{\tau_j = j\Delta\tau,\ j = 0, 1, 2, \ldots, M\}$, by using the standard Euler method of summing increments of the process $U(\tau)$, we get

$$\begin{aligned} &U(\tau_0) = 0, \\ &U(\tau_j) = U(\tau_{j-1}) + \Delta\tau^{1/\alpha}\varepsilon_j \end{aligned} \tag{14}$$

where ε_j are the i.i.d totally skewed positive α-stable random variables. The procedure of generating ε_j is as follows:

$$\varepsilon_j = c_1 \frac{\sin[\alpha(V + c_2)]}{[\cos(V)]^{1/\alpha}} \left(\frac{\cos[V - \alpha(V + c_2)]}{W}\right)^{(1-\alpha)/\alpha} \tag{15}$$

where $c_1 = [\cos(\pi\alpha/2)]^{-1/\alpha}$, $c_2 = \pi/2$, the random variable V is uniformly distributed on $(-\pi/2, \pi/2)$ and W has exponential with mean one. The iteration (6) ends when $U(\tau)$ crosses the level T, i.e. when for some $j_0 = M$ we can get $U(T_{M-1}) \leqslant T < U(T_M)$. Now for every element t_i of the lattice, we can find the element τ_j such that $U(T_{j-1}) \leqslant t_i < U(T_j)$, and finally from definition (5), we get that in such a case

$$X\left(S_{\mathrm{t_i}}\right) = \tau_j. \tag{16}$$

Since $U(\tau)$ is strictly increasing, M always exists, and the above procedure can be implemented efficiently.

(II) In the second step, we find the approximation $X(S_{t0}), X(S_{t1}), \ldots, X(S_{tN})$ of the subordinated process $X(S_t)$. We approximate the solution of $X(\tau)$ of the SDE (6) on the lattice $\{\lambda_k = k\Delta\lambda\colon k = 0, 1, \ldots, L\}$. The number L is equal to the first integer that exceeds the value $S_{tN}/\Delta\lambda$. Employing the classical Euler Maruyama method, we get

$$\begin{aligned} &X(\lambda_0) = 0 \\ &X(\lambda_k) = X(\lambda_{k-1}) + V'(X(\lambda_{k-1}),\quad U(\lambda_{k-1}))\eta^{-1} + (2D)^{1/2}\Delta\lambda^{1/2}\xi_k. \end{aligned} \tag{17}$$

$U(\tau)$ on the lattice $\{\lambda_k = k\Delta\lambda\colon k = 0, 1, \ldots, L\}$ can be approximated by the values of $U(\tau)$, which are obtained in step 1. In the case for some index m, the condition $\tau_m < \lambda_k < \tau_{m+1}$ holds true, we get

$$U(\lambda_k) = U(\tau_m). \tag{18}$$

(III) To obtain the value of subordinator $S_{t0}, S_{t1}, S_{t2} \ldots, S_{tN}$ on the lattice τ_j, we can use the value of λ_k on the lattice $\{\lambda_k = k\Delta\lambda\colon\ k = 0, 1, \ldots, L\}$ to approximate S_t by finding for every t_i from the lattice $\{t_j = j\,\Delta t\colon\ j = 0, 1, 2, \ldots, N\}$ an index k such that the condition $\lambda_k \leqslant S_{ti} \leqslant \lambda_{k+1}$ holds true. Then we get

$$X(S_{t_i}) = X(\lambda_k). \tag{19}$$

Since the subordination process $X(\tau)$ is not continuous for $0 < \mu < 2$, linear interpolation is not used at this point. The procedure of generating realizations of random variable ξ_k with symmetric or skewed Lévy noise can be described as follows:

$$\xi_k = c_1 \frac{\sin[\mu(V + c_2)]}{[\cos(V)]^{1/\mu}} \times \left[\frac{\cos(V - \mu(V + c_2))}{W}\right]^{(1-\mu)/\mu} \tag{20}$$

with

$$c_1 = \left(1 + \beta^2 \tan^2\left(\frac{\pi\mu}{2}\right)\right)^{1/2\mu} \tag{21}$$

$$c_2 = \arctan\left(\beta \tan\frac{\pi\mu}{2}\right)\Big/ \mu \tag{22}$$

where the random variable V is uniformly distributed on $(-\pi/2, \pi/2)$ and W has exponential distribution with mean one.

In the Euler scheme, we should note that in the case of impulsive noise the numerical integration path may escape to infinity rapidly with the decrease of the stability index μ. To eliminate the problem, we impose an constraint on the value of the process $X(\tau)$, which allows us to integrate equation (17) with any time step length [19, 20]. For example, we put $X(\lambda_k) = 10$ when $X(\lambda_k) > 10$ and put $X(\lambda_k) = -10$ when $X(\lambda_k) < -10$ in the dynamics with saturation effects. Recently, Heinsalu and Hänggi *et al* [21] presented a numerical method to generating subdiffusive dynamics via underlying CTRW. In their approach, it is necessary to generate successive residence times of the particle, which are Mittag–Leffler distributed. The authors stated that one can conveniently use the Pareto law to replace the Mittag–Leffler distribution when the subdiffusive index α is far from one. However, as the subdiffusive index α approaches one, the use of the Mittag–Leffler distribution, which precisely matches the Fokker–Planck description, should be used preferably. In the present paper, for both the subdiffusive and superdiffusive cases, the subordination technique provides a natural way to study the SR phenomenon.

3. Results

As discussed above, the subordination process is integrated by means of Euler's method. We choose the bistable system as $a = 2$ and $b = 1$, and adopt an integration step of $\Delta\lambda = 10^{-3}$. The simulations are repeated 250 times for each parameter set and the SNR is computed by recourse to the average power spectral density. Sample paths of anomalous diffusion with the parameters of $\alpha = 0.7$, $\mu = 1.7$ are plotted in figure 1. The interplay between subdiffusion and Lévy noise is observed. The subdiffusive behaviour of the system is caused by the inverse-time-α-stable subordinator, and Lévy-type jump is inherited from the parent process in equation (6).

When the superdiffusive index μ is fixed, we examine the SNR performance as a function of the noise intensity D, for different values of subdiffusive index α. Figure 2 illustrates that the SNR first increases at first, then decreases, and exhibits a nonmonotonic dependence on the noise intensity D. When $\alpha = 0.9$, a pronounced peak can be found at the optimal noise value $D_{\mathrm{SR}} = 0.18$. The optimal noise value D_{SR} only shifts slightly as the subdiffusive index

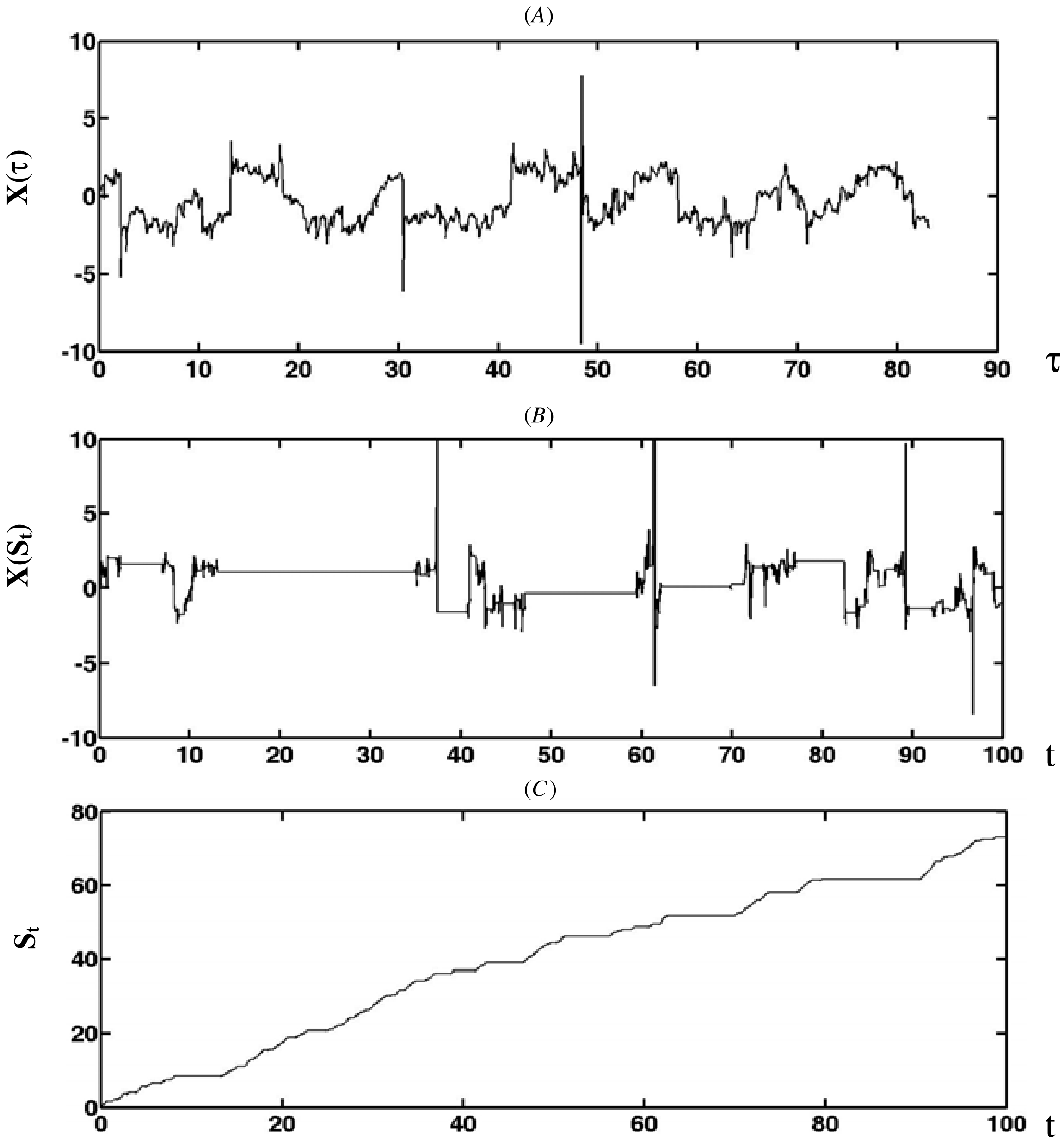

Figure 1. Exemplary trajectories of (A) the parent $X(\tau)$ (B) anomalous diffusion $X(S_t)$ (C) the subordinator S_t in the presence of the bistable system with an input of sinusoidal signal. The parameters are $\alpha = 0.7$, $\mu = 1.7$, the time step of the integration $\Delta\lambda = 0.001$, $a = 2$, $b = 1$ and the driving frequency $Q = 0.1$. The constant intervals indicate the long waiting time induced by the subdiffusive character.

α varies. When the index α is smaller than 0.7, the SNR decreases very slowly after the noise intensity D reaches the optimal value D_{SR}, which indicates that the SNR measure becomes insensitive to the variance of noise for smaller subdiffusive index α. It can be explained that

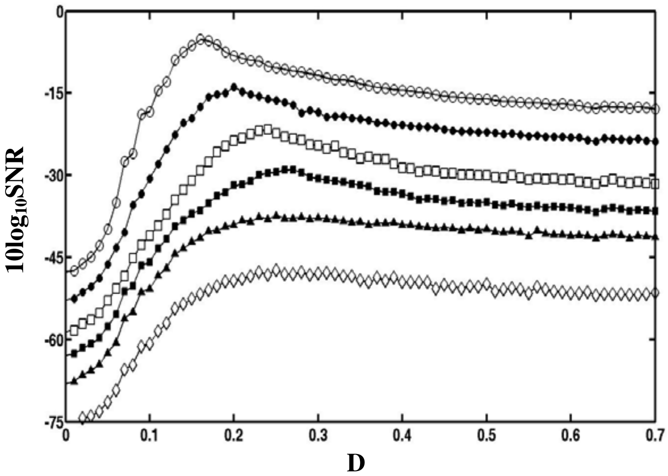

Figure 2. SNR versus noise intensity D for different values of subdiffusive index α. The parameters are $\mu = 1.7$, $a = 1$, $b = 1$ and $Q = 0.1$, skewness index $\beta = 0$. The different curves represent the SNR values with different index α, showing a decrease of the subdiffusive response to the external signal. In particular, it is shown: $\alpha = 0.9$ (O), $\alpha = 0.8$ (●), $\alpha = 0.7$ (□), $\alpha = 0.6$ (■), $\alpha = 0.5$ (▲), $\alpha = 0.4$ (◇).

noise plays a minor role as longer intervals induced by subdiffusive character occur. We also observe that with the decrease of the subdiffusive index α, the maximum of the SNR continuously decays and the nonmonotonic behaviour vanishes. The results show that the SR effect is weakened as the subdiffusive character is enhanced. We compare the results with the analytical results obtained by Liu and So in the pure subdiffusive case, and show that they are qualitatively consistent.

Based on the viewpoint of subordination, one possible explanation for the competitive case can be given as follows. In a bistable system, the best time to switch between potentials is when the relevant potential barrier assumes a minimum. This is the case when the potential $V(x, \tau) = V(x) - \mathrm{A}_0 \times \cos(Q\tau + \varphi)$ is tilted most extremely to the right or the left. If the system switches at this time into the other well, then it takes half one periodic waiting time into the other well until the new relevant barrier assumes a minimum. If the system misses the best opportunity to jump, it has to wait another full period until the relevant barrier for a switch assumes the minimum again. However, for the subordination process, the time-dependent force only changes in real time t. As indicated above, before the subordination $X(S_t)$, the actual force is given by $F(x, U(\tau))$. By replacing the $V(x, t)$ with increment time $U(\tau)$, we get $V(x, U(\tau)) = V(x) - \mathrm{A}_0 \times \cos(QU(\tau) + \varphi)$. We show that the periodic character of the time-dependent force is diminished. As the subdiffusive character is enhanced, i.e. the increment level of $U(\tau)$ is increased, the non-periodic property of the bistable system becomes more pronounced, and input–output synchronization is weakened. Actually, before the subordination process is carried out, the noise-induced phase synchronization has occurred, and the subordinator S_t only plays a role in changing the time scale of time-dependent force from random time τ to actual time t. That is to say, the increment process $U(\tau)$ will weaken the noise-induced synchronization, and the subordinator S_t causes the long rest induced by subdiffusion to be a reality.

When the subdiffusive index α is fixed, the dependence of the SNR on noise is plotted for different values of superdiffusive index μ in figure 3. It can be seen that the SNR increases

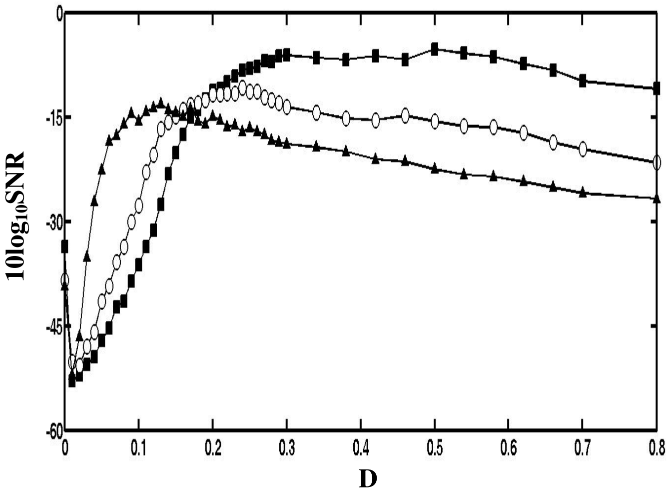

Figure 3. SNR versus symmetric noise intensity D. The parameters are $a = 1$, $b = 1$, $Q = 0.1$, stability index $\alpha = 0.8$ and skewness index $\beta = 0$. The different curves are plotted for different values of the index μ. In particular, it is shown: $\mu = 1.9$ (■), $\mu = 1.7$ (O), $\mu = 1.5$ (▲).

with increasing noise intensity D at first, then decreases. The SNR curve still exhibits a nonmonotonic SR behaviour. As the superdiffusive index μ is decreased, the tails on the noise bell curves become thicker and the infinite variance noise becomes more impulsive. We observe that the optimal value D_{SR} of noise intensity decreases, i.e. $D_{\mathrm{SR}} = 0.31$, $D_{\mathrm{SR}} = 0.22$, and $D_{\mathrm{SR}} = 0.13$, for $\mu = 1.9$, $\mu = 1.7$, and $\mu = 1.5$ respectively. The maximum of the SNR becomes lower and the bell-shaped SNR curve becomes narrower, i.e. the lower the exponent μ, the lower the SNR peak value. It can be explained as follows from three cases respectively.

The dynamics property of Lévy flight in a harmonic potential should be reviewed first. Actually, despite the infinite variance of Lévy noise, we argue that the output of the bistable system should have finite variance [11]. There exists a critical value C_r,

$$C_r = 4 - \mu, \tag{23}$$

when the function of the potential $U(x)$ asymptotic dependence on x has the form $U(x) \sim |x|^C$. If the condition $C > C_r$ holds true, the system output should have finite variance. When $D < D_{\mathrm{SR}}$, the switching events in a bistable system are very rare, thus the periodic components of the output signal are determined primarily by intrawell motion, in which the periodic components of interwell dynamics are weak. However, when the noise intensity D is fixed, impulsive noise can make the switching events between the two states become more frequent as the superdiffusive index μ is decreased. The impulsive noise helps the particle to jump across the barrier. Hence, the periodic components of interwell dynamics are enhanced. Before the noise intensity D reaches the optimal value D_{SR}, the SNR value will become larger as the superdiffusive index μ is decreased. In the case when $D = D_{\mathrm{SR}}$, the synchronization effects induced by impulsive noise are weaker than those effects which are induced by finite-variance noise such as Gaussian noise. The long jump induced by the heavy-tailed distribution can slightly diminish the periodic components of interwell dynamics. As the superdiffusive index μ is decreased, the optimal SNR will become lower. In the case when the condition $D > D_{\mathrm{SR}}$ holds true, a loss of synchronization starts to occur. Compared with Gaussian noise at

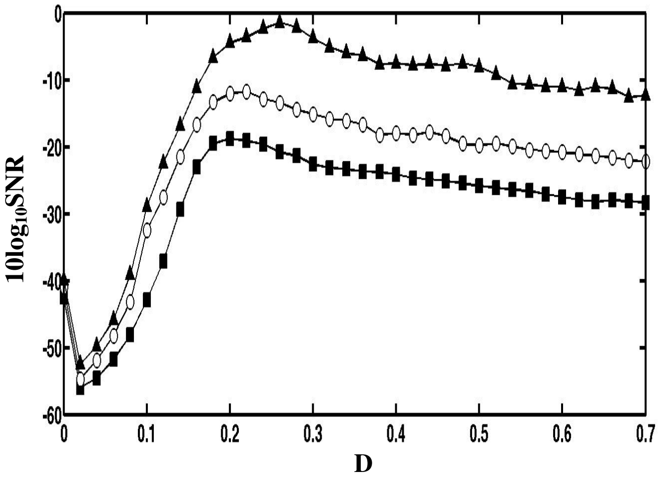

Figure 4. SNR versus noise intensity D for different values of driving frequency Q. The parameters are $\mu = 1.7$, $\alpha = 0.8$, $a = 1$, $b = 1$, skewness index $\beta = 0$. The different curves represent different SNR values with different frequency Q. It is shown: $Q = 1$ (▲), $Q = 0.1$ (O), $Q = 0.01$ (■).

the same intensity D, the impulsive noise makes the system flip too many times between two wells, and the periodic components of interwell dynamics are greatly weakened. The SNR becomes lower as the superdiffusive index μ is decreased.

The SNR versus the noise intensity D at different driving frequencies is depicted in figure 4. For a fixed angular modulation frequency Q, a nonmonotonic behaviour of SNR versus the noise D is shown. With the decreasing driving frequency, we note that the maximum of the SNR decreases and the peak positions shift to smaller noise intensity. A frequency-dependence phenomenon is observed. As a matter of fact, any deviations of residence time distribution from strictly exponential form indicate a deviation from Markovian behaviour. According to the non-Markovian SR theory [22], the long jump induced by superdiffusion does not change the Markovian nature of the bistable dynamics. Due to the existence of subdiffusion, we conclude that non-exponential relaxation time or long rest lead the bistable system to the deviation from the Markovian case and the frequency-dependence phenomenon can occur. This is a main difference from classical two-state SR theory [1].

In the foregoing discussions, the noise intensity D is tunable, and the SR phenomenon is called noise-induced SR. When fixing noise intensity, we also study the parameter-induced SR effect by tuning the system parameter b. The SNR dependence on the parameter b for different values of superdiffusive index μ is plotted in figure 5. It is shown that parameter-induced SR still occurs in the competitive case. With the decrease of subdiffusive index α, the parameter-induced SR effect is diminished. However, as the superdiffusive index μ is decreased, we observe that the SNR curve becomes narrower and the peak value becomes larger. In contrast with noise-induced SR, it is shown that the SNR performance can be improved by tuning the system parameters. When the superdiffusive index $\mu = 1.5$, the SNR value even becomes positive. The results indicate that in a certain range the subdiffusive effect can be canceled out by improving the superdiffusive character with optimal system parameters. The decrease of the periodic components of interwell motion induced by the heavy-tailed waiting time can sometimes be recovered by long jumps induced by Lévy flights.

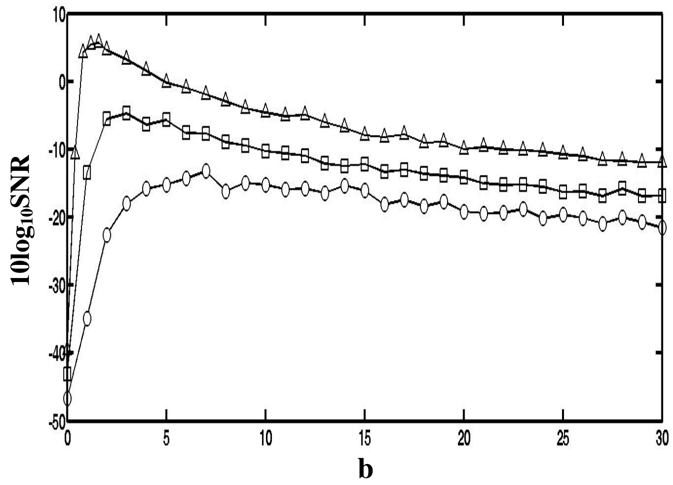

Figure 5. SNR versus system parameter b. The parameters are $a = 1$, $Q = 0.1$, stability index $\alpha = 0.8$, and skewness index $\beta = 0$. The different curves are plotted with different values of superdiffusive index μ. In particular, it is shown: $\mu = 1.9$ (O), $\mu = 1.7$ ($\square$), $\mu = 1.5$ ($\triangle$).

4. Conclusions

We have analysed the SR effect of the subdiffusive bistable model driven by Lévy noise, with a sine input. Such a system, when both superdiffusion and subdiffusion are present, has a known form of the FFPE. However, in the general case where external force is time-space-dependent, we have not been able to find a valid form of the FFPE and have to resort to an analysis of numerical simulations based on the subordination technique [15].

Through the numerical approach we have studied the dependence of the system response on different system parameters. From the above results, it is apparent that the enhancement of the output SNR versus the noise intensity D is decreased with increasing subdiffusive character. We conclude that the phenomenon of the weakness of the SNR is robust in very general systems where superdiffusion and subdiffusion coexist. It is seen that the enhancement of the output SNR versus the noise intensity D is decreased when the superdiffusive index is decreased. The result indicates that the impulsive character of Lévy noise clearly contributes to diminishing the SR phenomenon in the subdiffusive system. For the competitive case, the SNR versus the noise intensity D is tested against different values of driving frequency. A frequency-dependence phenomenon is shown. Finally, we find that the enhancement of the output SNR versus the system parameters is improved when the superdiffusive index is decreased. We conclude that the decrease of the superdiffusive index in a certain range can enhance the parameter-induced SR effect with fixed subdiffusive character, i.e. the subdiffusive effect can sometimes be canceled out by increasing superdiffusive character.

One aspect worth studying in detail is the residence time distribution of the system, which provides another possibility to characterize SR. We hope to extend the application of the subordination technique to consider the noise-induced phase synchronization mechanism of the bistable system in the competitive case between long rests and long jumps [23]. This problem will be the subject of further work.

Acknowledgments

The authors are very grateful to the editor and the anonymous referees whose suggestions have immensely improved the quality of the manuscript. We thank M Magdziarz for many constructive suggestions about the subordination technique and simulating process. We greatly appreciate financial support provided by the fellowship from Institute for Pure and Applied Mathematics at UCLA where the work was carried out. The work is partly sponsored by the National Science Foundation of China (no. 60775057).

References

[1] Gammaitoni L, Hänggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223–85
[2] Gammaitoni L, Hänggi P, Jung P and Marchesoni F 2009 *Eur. Phys. J.* B **69** 1–3
[3] Hänggi P, Jung P, Zerbe C and Moss F 1993 *J. Stat. Phys.* **70** 229–45
[4] Goychuk I and Hänggi P 2003 *Phys. Rev. Lett.* **91** 070601
[5] Yim M Y and Liu K L 2006 *Physica* A **369** 329–42
[6] Zeng L, Xu B and Li J 2007 *Phys. Lett.* A **361** 455–9
[7] Dybiec B and Gudowska-Nowak E 2006 *Acta Phys. Pol.* B **37** 1479–90
[8] Zeng L, Bao R and Xu B 2007 *J. Phys. A: Math. Theor.* **40** 7175–85
[9] Denisov S I, Horsthemke W and Hänggi P 2009 *Eur. Phys. J.* B **68** 567–75
[10] Sokolov I M, Heinsalu E, Hänggi P and Goychuk I 2009 *Europhys. Lett.* **86** 30009
[11] Metzler R and Klafter J 2000 *Phys. Rep.* **339** 1–77
[12] So F and Liu K L 2000 *Physica* A **277** 335–48
[13] Heinsalu E, Patriarca M, Goychuk I and Hänggi P 2007 *Phys. Rev. Lett.* **99** 120602
[14] Heinsalu E, Patriarca M, Goychuk I and Hänggi P 2009 *Phys. Rev.* E **79** 041137
[15] Magdziarz M, Weron A and Klafter J 2008 *Phys. Rev. Lett.* **101** 210601
[16] Magdziarz M and Weron A 2007 *Phys. Rev.* E **75** 056702
[17] Jung P and Hänggi P 1989 *Europhys. Lett.* **8** 505–10
[18] Jung P and Hänggi P 1991 *Phys. Rev.* A **44** 8032–42
[19] Mitaim S and Kosko B 2004 *IEEE Trans. Neural Netw.* **15** 1526–40
[20] Kosko B and Mitaim S 2003 *Neural Netw.* **16** 755–61
[21] Heinsalu E, Patriarca M, Goychuk I, Schmid G and Hänggi P 2006 *Phys. Rev.* E **73** 046133
[22] Goychuk I and Hänggi P 2004 *Phys. Rev.* E **69** 021104
[23] Casado Pascual J, Gómez Ordóñez J, Morillo M, Lehmann J, Goychuk I and Hänggi P 2005 *Phys. Rev.* E **71** 011101